

# **A Graphical Approach for Solving Three-Variable Linear Programming Problems**

**T.R. Kannan, G. Dinakaran, N.J. Lavanya**  
Mechanical Engineering Department  
K.L.N. College of Engineering, Pottapalayam- 630611.

## **Abstract**

A linear programming problem (LP) deals with determining optimal allocations of limited resources to meet the given objectives. The objectives are usually maximizing profit, minimizing total cost, maximizing utility etc. In general, LP problems are solved by two methods. One is algebraic method (Simplex method) and the other one is graphical method. LP problems that involve only two variables can be solved by both methods. LP problems having three variables can also be solved graphically but is more difficult when tried out manually. Solving three-variable problems by simplex method is also tedious and time consuming. In the present work, a methodology for solving three variable LP problems using graphical method is developed. This method is automated using AutoCAD with Visual Basic Application (VBA). A mathematical approach for solving the problem is also developed.

## **Introduction**

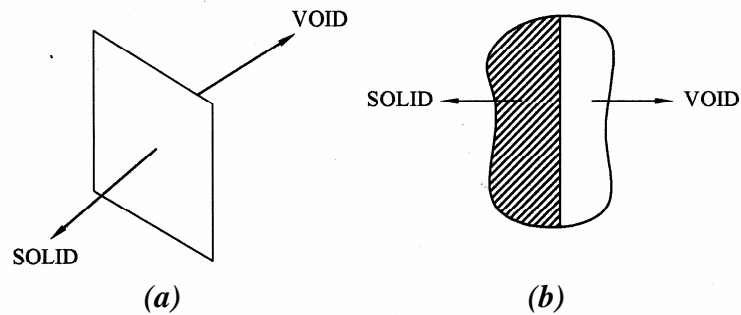
Most of the real life problems have numerous constraints. But a majority of the constraints are weak constraints and they only have a minimal effect on the performance of the system. Hence, such constraints are normally neglected during modeling. In general, a large number of problems can be modelled into three variable problems involving only the predominant constraints. Even though the three-variable problems can be solved using simplex method, the calculations are cumbersome. In this paper the graphical method, which is normally used to solve the two-variable problem, is extended to solve three-variable problems. For optimizing the objective function, this methodology obtains the vertices of a 3D solid that is bounded by faces, which represent the constraints and non-negativity constraints. A mathematical implementation of this graphical method without using AutoCAD is also developed. Both procedures are explained in detail in the following sections.

## **Graphical Method for three-variable problem**

Given a three-variable problem to optimize, subject to some constraints and non-negativity constraints, the methodology for solving the problem using graphical method is as follows. The constraint equations and non-negativity constraints are first represented as half-spaces. Then a 3D solid that is obtained by considering these half-spaces as boundaries is obtained. The optimal solution lies in the vertices (corner points) of this 3D solid. The coordinates of the vertices are applied in the objective function and the optimal solution is determined. This can be explained clearly with an example.

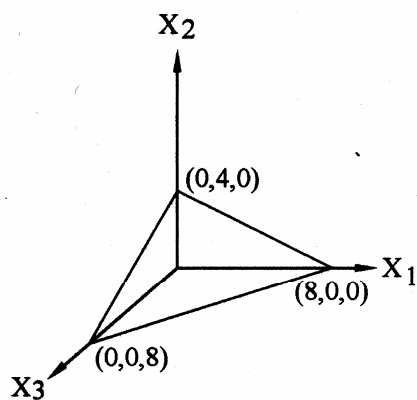
Consider a maximization problem

$$\begin{aligned}
 Z &= x_1 + 4x_2 + 5x_3 && \text{subject to constraints} \\
 3x_1 + 6x_2 + 3x_3 &\leq 24 && (1) \\
 x_1 + 2x_2 + 3x_3 &\leq 12 && (2) \\
 3x_1 + 2x_2 + x_3 &\geq 6 && (3) \\
 \text{and } x_1, x_2, x_3 &\geq 0 && (4)
 \end{aligned}$$

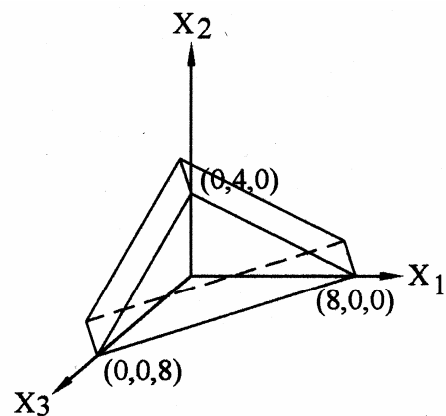


**Fig 1. Typical Half-spaces depicting a surface dividing the 3D space into solid and void region**

We know that a plane equation can be represented as  $ax_1 + bx_2 + cx_3 = d$ . As the constraint equations are also of this type, the constraint equations can be thought of as planes. Since the constraints are of inequality type, they can be thought of as half-spaces. A half-space consists of a surface (usually of infinite extent), which completely divides the three dimensional space into a solid region and a void region as shown in Fig 1. Any point that satisfies the constraint equation lies in the solid region.



**Fig 2. Representation of the constraint equation 1 as a plane**



**Fig 3. Extrusion of the constraint plane towards origin to represent the plane as a half-space**

In order to represent a half-space, the points that lie on the plane (say intercepts) are first determined. By substituting the values of any two variables as zero, the  $x_1, x_2, x_3$  intercepts of the plane can be determined. For equation 1, by substituting  $x_2 = 0$  and  $x_3 = 0$ , we get  $x_1 = 8$  (Fig 2).

Similarly, when  $x_1 = 0$  and  $x_3 = 0$ , we get  $x_2 = 4$  and when  $x_1 = 0$  and  $x_2 = 0$ , we get  $x_3 = 8$ . Thus the three intercept-coordinates are obtained. They are  $(8,0,0)$ ,  $(0,4,0)$  and  $(0,0,8)$  for the plane represented by equation 1 as shown in Fig 2. Then, the plane is extruded along the solid side in order to represent it as a half-space. The solid side is determined as follows. On substituting  $(0,0,0)$  in the constraint equation, if the resultant satisfies the inequality, then the plane is extruded about the face normal that points towards the origin as shown in Fig 3. On substituting  $(0,0,0)$  in the equation 1, the LHS is 0, which is lesser than the RHS (i.e. 24). Hence for this constraint, the plane has to be extruded towards the origin about the face normal. If the resultant does not satisfy the inequality, extrusion has to be carried out in an opposite direction. Theoretically half-spaces have infinite solid region. Since generating an infinite half-space is not practical, the planes may be extruded only by a finite length  $K$ . The value of  $K$  is assumed as 100. The user can also give any other value for  $K$  provided that the  $K$  value is sufficient to include the feasible solution space. The planes that divide the 3D space into solid and void region has to be an infinite plane. Instead, a finite plane with the intercepts as the vertices are first generated. Then these planes are scaled about the centre point of the plane. Scaling has to be done only for constraints having non-zero coefficients for all the three variables.

In order to find out the face normal, the centre of the plane  $(X_c, Y_c, Z_c)$  are determined first. The centre coordinates can be obtained by

$$X_c = \frac{a}{3}; \quad Y_c = \frac{b}{3}; \quad Z_c = \frac{c}{3}$$

where  $a, b, c$  are the  $x_1, x_2, x_3$  intercepts of the plane as shown in Fig 4.

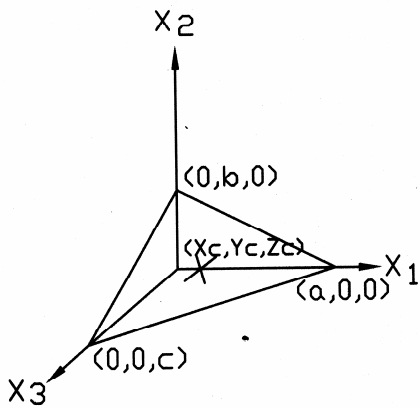
The coordinates of a point  $(X_n, Y_n, Z_n)$  lying at a distance of  $K$  in the face normal is determined by

$$\frac{X_n - X_c}{L} = \frac{Y_n - Y_c}{M} = \frac{Z_n - Z_c}{N} = K$$

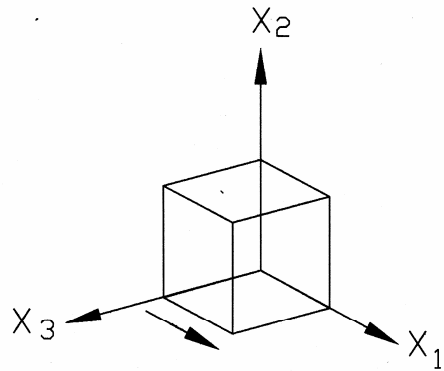
$$X_n = KL + X_c; \quad Y_n = KM + Y_c; \quad Z_n = KN + Z_c;$$

where  $L, M, N$  are given by

$$L = \frac{(b * c)}{\sqrt{(a^2 b^2) + (b^2 c^2) + (c^2 a^2)}}; \quad M = \frac{(c * a)}{\sqrt{(a^2 b^2) + (b^2 c^2) + (c^2 a^2)}}; \quad N = \frac{(a * b)}{\sqrt{(a^2 b^2) + (b^2 c^2) + (c^2 a^2)}}$$



**Fig 4. Centre point of a plane**



**Fig 5. Representation of  $x_1 \geq 0$  as a half-space**

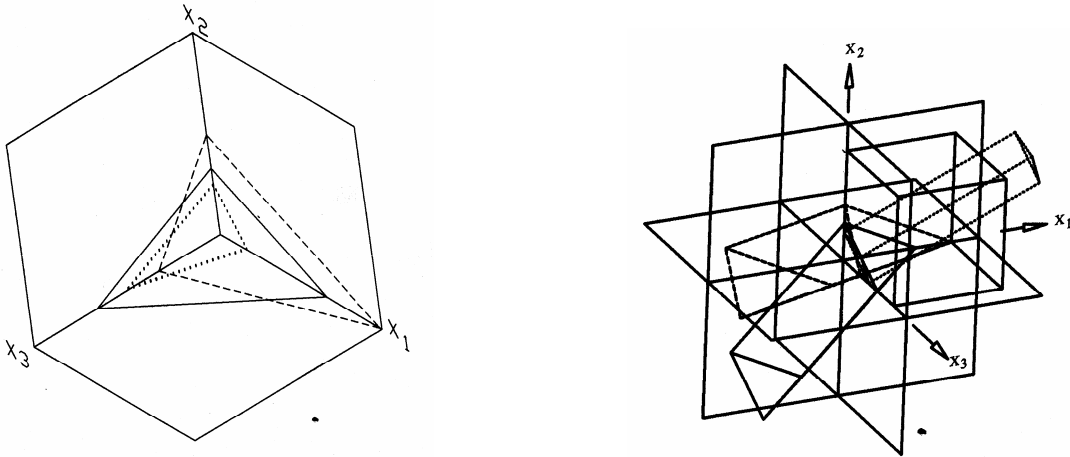
The value of  $K$  depends on two factors. They are the octant in which the plane is lying and whether the extrusion has to be carried out towards the origin or away from the origin. The values of  $K$  for planes lying in different octants with different direction of extrusion are given in Table 1.

**Table 1.  $K$  values for different octants**

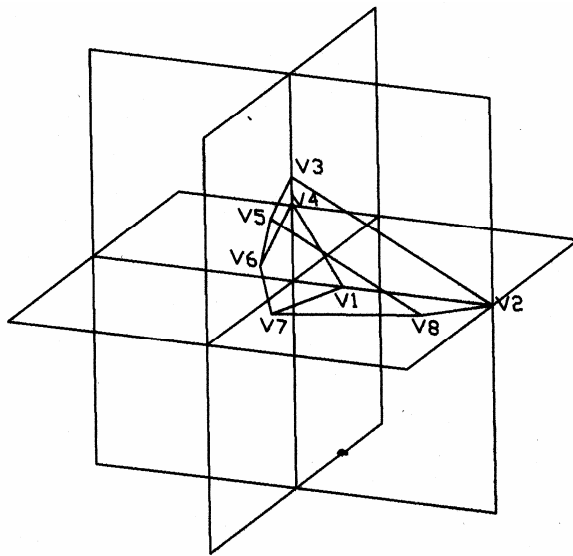
Octant	Axes	Value of $K$	
		Towards Origin	Away from the Origin
I	+X1, +X2, +X3	-100	100
II	-X1, +X2, +X3	100	-100
III	-X1, -X2, +X3	-100	100
IV	+X1, -X2, +X3	100	-100
V	+X1, -X2, -X3	-100	100
VI	+X1, +X2, -X3	100	-100
VII	-X1, +X2, -X3	-100	100
VIII	-X1, -X2, -X3	100	-100

The non-negativity constraints (i.e.  $x_1, x_2, x_3 \geq 0$ ) are also represented as three half-spaces. To represent  $x_1 \geq 0$ ,  $x_2x_3$  plane is extruded along the +ve  $x_1$  axis as shown in Fig 5. Similarly to represent  $x_2 \geq 0$  and  $x_3 \geq 0$ ,  $x_1x_3$  and  $x_1x_2$  planes are extruded along +ve  $x_2$  and +ve  $x_3$  axes respectively. Thus, all the constraints and non-negativity constraints are represented as half-spaces as explained earlier. The half-space representation of all the constraints and the non-negativity constraints for the example problem is shown in Fig 6. After representing all the constraints as half-spaces, the solid obtained by the intersection of all these half-spaces is extracted. For the example problem shown above, on intersection of all the half-spaces, a solid as

shown in Fig 7 is obtained. This solid represents the feasible solution space i.e. the point that lies on/within this solid is a feasible solution. All the vertices (corner points) of the solid are substituted in the objective function and the optimum value among these feasible solutions is determined. The vertices and the corresponding objective function values are shown in Table 2. The optimum value of the objective function is 22 and the corresponding variable values are (0,3,2).



**Fig 6. (a) Plane representation of the constraints  
(b) Half-space representation of the constraints**

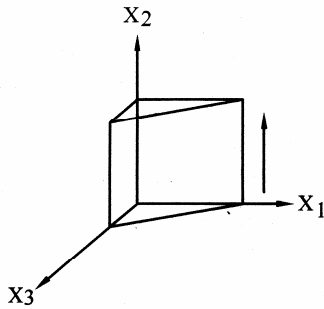


**Fig 7. Solid obtained on intersection of all the constraints [feasible solution space]**

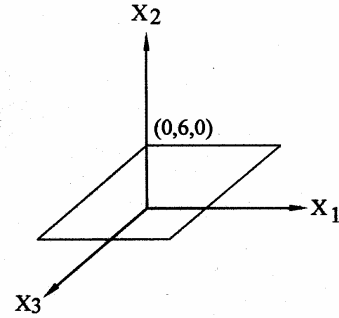
**Table 2. Feasible solution set and the respective objective function values.**

S.No.	Vertices	Coordinates	Objective Function value (Z)
1.	V1	(2,0,0)	2
2.	V2	(8,0,0)	8
3.	V3	(0,4,0)	16
4.	V4	(0,3,0)	12
5.	<b>V5</b>	<b>(0,3,2)</b>	<b>22</b>
6.	V6	(0,1.5,3)	21
7.	V7	(7.5,0,3.75)	19.5
8.	V8	(6,0,2)	16

For LP problems with constraints having one coefficient as zero, (i.e. say  $x_1 + 0x_2 + 3x_3 \leq 12$ ) the plane is defined by (12,0,0), (0,0,0) and (0,0,4) and extruded towards +ve  $x_2$  as shown in Fig 8. For constraints having any two coefficients as zero (say  $x_2 \leq 6$ ), the constraint is represented by a plane parallel to  $x_1x_3$  plane intercepting  $x_2$  axis at 6 as shown in Fig 9. For equality constraint, the plane is not extruded.



*Fig 8 Half-space representation for constraints having  $x_2$  coefficient as zero*



*Fig 9 Half-space representation for constraints having  $x_1$  and  $x_3$  coefficient as zero*

### **AutoCAD implementation of the graphical method**

The graphical method is automated using VBA. VBA is an application development language that comes as a pack with AutoCAD by which the AutoCAD objects can be directly manipulated. AutoCAD VBA permits the VBA environment to run simultaneously with AutoCAD and provides programmatic control of AutoCAD through ActiveX automation interface.

The application is made as an icon in the AutoCAD screen. On clicking the icon, an input dialog box appears. The objective function and the constraints have to be typed in and on clicking the OK button the optimal function value along with the optimal variable values are displayed in a dialog box. On getting the inputs, the application calculates the intercepts of each constraint and generates a 3D face in the AutoCAD screen and extrudes it in the direction deduced as explained earlier. Then it generates the intersection solid and extracts the vertices of the solid. From the coordinates of these vertices, the application calculates the optimal solution and displays it on the screen.

### **Mathematical Implementation of the graphical method**

The graphical method explained in the previous section was implemented in AutoCAD with VBA. The same procedure can also be implemented mathematically without the use of any 3D modeling software. In mathematical method, instead of extruding the planes to get a bounded solid, the point of intersection of different planes are found out first. Using the coordinates of these points, the objective function is evaluated and the optimum solution among these solutions is found out. The  $x_1$ ,  $x_2$ ,  $x_3$  intercepts of the each plane are found out first. Finding out the intercept coordinates has already been explained in the previous section. Then, the point of intersection of consecutive constraint planes with  $x_1x_2$ ,  $x_2x_3$ ,  $x_1x_3$  planes is determined.

Finally the point of intersection between all the constraint planes (other than the non-negativity constraints) is determined. Given three planes represented by

$$a_1x_1 + b_1x_2 + c_1x_3 = d_1; \quad a_2x_1 + b_2x_2 + c_2x_3 = d_2; \quad a_3x_1 + b_3x_2 + c_3x_3 = d_3;$$

The point of intersection  $(x_1, x_2, x_3)$  between any three planes can be determined by

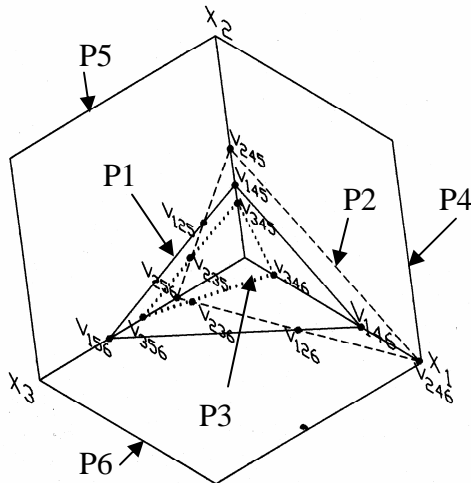
$$\frac{x_1}{\Delta_1} = \frac{-x_2}{\Delta_2} = \frac{x_3}{\Delta_3} = \frac{-1}{\Delta_4}$$

where  $\Delta_1, \Delta_2, \Delta_3, \Delta_4$  are given by

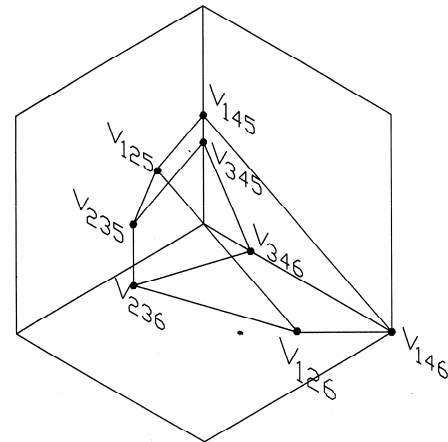
$$\Delta_1 = \begin{vmatrix} b_1 & c_1 & -d_1 \\ b_2 & c_2 & -d_2 \\ b_3 & c_3 & -d_3 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} c_1 & -d_1 & a_1 \\ c_2 & -d_2 & a_2 \\ c_3 & -d_3 & a_3 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} -d_1 & a_1 & b_1 \\ -d_2 & a_2 & b_2 \\ -d_3 & a_3 & b_3 \end{vmatrix}; \quad \Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If  $\Delta_4 = 0$ , then there is no intersection

For the example problem given in the earlier section, the constraint equations 1, 2 and 3 are shown as planes P1, P2 and P3 in solid line, hidden line and dotted lines respectively in Fig 10. The non-negativity constraints  $x_1, x_2, x_3 \geq 0$  are shown as planes P4, P5 and P6. The intersection points of different planes are shown as  $V_{ijk}$ , where  $V_{ijk}$  is the point obtained by the intersection of planes  $P_i, P_j, P_k$ . We can see from these figures that all the intersection points obtained do not lie on the solution space solid (say  $V_{156}, V_{246}, V_{245}$ , etc.,). Hence such points have to be removed from the solution set as these points do not satisfy some of the constraint equations. These points are identified by checking whether any point does not satisfy any of the constraint equations. For example, the coordinates of the point  $V_{156}$  are  $(0,0,8)$ . The coordinates of  $V_{156}$  does not satisfy the constraint equation 3 (i.e. the LHS becomes greater than the RHS). Hence, this point is removed from the solution set. Then the points having negative coordinate values are also neglected, as such points do not satisfy the non-negativity constraints. After removing the unwanted points, the remaining points form a solution space among which the optimum solution is available. The solution space solid and the bounding vertices ( $V_{145}, V_{345}, V_{346}, V_{146}, V_{126}, V_{236}, V_{235}, V_{125}$ ) are shown in Fig 11. In this method also, the same optimum solution as that in the graphical method is obtained. The procedure explained above is implemented without the use of AutoCAD model space.



**Fig 10. Plane representation of the constraints**



**Fig 11. Solution space solid and the bounding vertices**

## **Results and Conclusion**

The above-proposed methods suit the variants of LP problems like alternate optima, infeasible solution, degeneracy and unbounded solution spaces with bounded solution. For infeasible solution LP problems, a bounded solid is not obtained in the case of graphical method, as all the half-spaces do not intersect to form a solid. In the case of mathematical method, even though some intersection points are obtained, no point satisfies all the constraint equations. In such case, it is concluded that there is no feasible solution. For degeneracy problems, cycling does not occur as in the case of simplex method. Instead, a solution is directly obtained as the redundant constraints that lead to degeneracy are automatically eliminated while getting an intersection solid. In mathematical method also, the points belonging to redundant constraints get automatically deleted as those points do not satisfy all the constraint equations. To find out whether a given LP problem has any other alternate optimal solutions, the values of the objective function obtained by substituting the co-ordinates of the bounded solid vertices are compared with one another. If the same objective function value is obtained for more than one vertex, then the problem has got alternate optima. The same procedure is followed in both the methods. For bounded solutions in unbounded solution space, both the method gives results. But for unbounded solutions in unbounded solution space, both methods do not suit.

The developed application software is tested with various LP problems. The results obtained are validated by comparing the results obtained from TORA and the results are found to be consistent. TORA is a software that solves LP problems by simplex method. The proposed methods effectively give the optimal solutions for all the LP problems.

## **References**

1. Operations Research, Hamdy A. Taha, Prentice Hall of India. 1999.
2. Operations Research, S. Dharani Venkatakrisnan. Keerthi Publishing House Ltd. 1999.
3. [www.autodesk.com](http://www.autodesk.com)